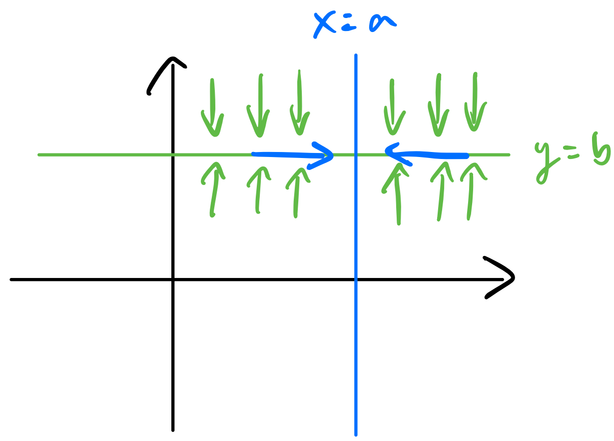
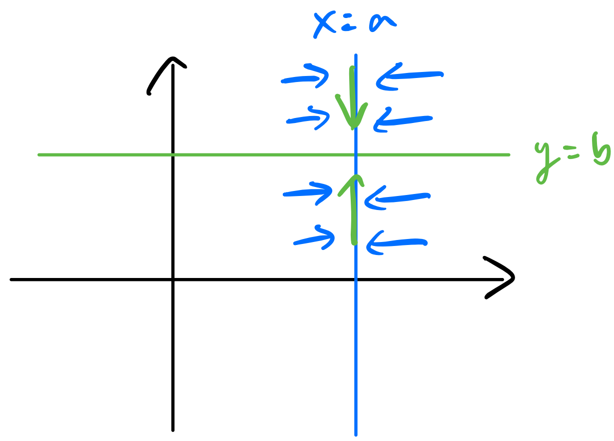
Iterated limits

$$\lim_{x \to a} \lim_{y \to b} f(x,y).$$



$$\lim_{y \to b} \lim_{x \to a} f(x,y).$$



There is no direct implication between:

(i) $\lim_{x \to a} \lim_{y \to b} f(x,y)$, $\lim_{y \to b} \lim_{x \to a} f(x,y)$ both exist and equal

(ii) $\lim_{(x,y) \to (a,b)} f(x,y)$ exist and equal

(i) $\not\Rightarrow$ (ii) :

$$f(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{else} \end{cases}, \quad (a,b) = (0,0)$$

Fixing $x \neq 0$,
$$\lim_{y\to 0} f(x,y) = 0$$

Fixing $y \neq 0$,
$$\lim_{x\to 0} f(x,y) = 0$$

Then $\lim_{x\to 0} \lim_{y\to 0} f(x,y) = 0$.

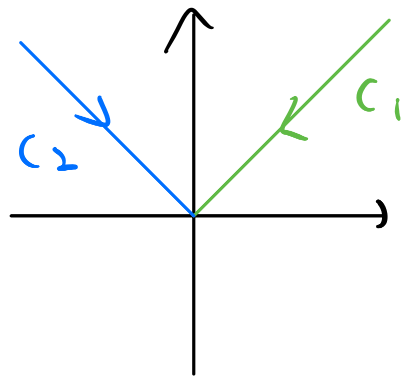
Then $\lim_{y\to 0} \lim_{x\to 0} f(x,y) = 0$.

On the other hand,

choosing paths :

C_1 : $x = y = t$, $t > 0$

C_2 : $x = -t$, $y = t$, $t > 0$.



$$\lim_{\substack{(x,y)\to(0,0) \\ \text{on } C_1}} f(x,y) = 1 \qquad \lim_{\substack{(x,y)\to(0,0) \\ \text{on } C_2}} f(x,y) = 0.$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ not exist.

(iii) $\not\Rightarrow$ (i)

$$f(x,y) = \begin{cases} x\cos\frac{1}{y} + y\cos\frac{1}{x} & , \text{ if } x,y \neq 0 \\ 0 & , \text{ if } x \text{ or } y = 0 . \end{cases}$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} r\left(\cos\theta\cos\frac{1}{r\sin\theta} + \sin\theta\cos\frac{1}{r\cos\theta}\right)$$

$$-2r \leq r\left(\cos\theta\cos\frac{1}{r\sin\theta} + \sin\theta\cos\frac{1}{r\cos\theta}\right) \leq 2r$$

Sandwich Thm

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ exists and $= 0$.

Fixing $x \neq 0$,

$$\lim_{y\to 0} x\cos\frac{1}{y} + y\cos\frac{1}{x}$$

choose $\{y_n\}_{n=1}^{\infty}$ where $y_n = \frac{1}{2n\pi}$

$\{y_n'\}_{n=1}^{\infty}$ where $y_n' = \frac{1}{(2n+1)\pi}$

$y_n, y_n' \to 0$ as

$$\lim_{n\to\infty} x\underbrace{\cos\frac{1}{y_n}}_{\equiv 1} + \underbrace{y_n\cos\frac{1}{x}}_{\to 0} = x$$

$$\lim_{n\to 0} \underbrace{x\cos\frac{1}{y_n'}}_{\equiv -1} + \underbrace{y_n'\cos\frac{1}{x}}_{\to 0} = -x$$

$$\Rightarrow \lim_{y\to 0} x\cos\frac{1}{y} + y\cos\frac{1}{x} \text{ not exist.}$$

As a result,

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \text{ not exist.}$$

same for

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

(§14.2)

Q4)

$$\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2$$

$$= \left(\frac{1}{2}+\frac{1}{-3}\right)^2 = \frac{1}{36}$$

$$\lim_{x\to 2}\lim_{y\to -3} \left(\frac{1}{x}+\frac{1}{y}\right)^2$$

$$= \lim_{x\to 2} \left(\frac{1}{x}-\frac{1}{3}\right)^2 = \frac{1}{36}$$

$$\lim_{y\to -3}\lim_{x\to 2} \left(\frac{1}{x}+\frac{1}{y}\right)^2$$

$$= \lim_{y\to -3} \left(\frac{1}{2}+\frac{1}{y}\right)^2 = \frac{1}{36}.$$

Q671. Continuously extend

$$f(x,y) = \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right) \text{ at } (0,0)$$

Continuous Definition :

$f: \mathbb{R}^n \to \mathbb{R}$ cont. at $\vec{a}$

if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$

Continuous extension :

for a function f not defined at some point $\vec{a}$,
e.g. the function in the question is not defined at $(0,0)$,

we define f at $\vec{a}$ s.t. f is continuous at $\vec{a}$.

i.e. $f(\vec{a}) := \lim_{\vec{x}\to\vec{a}} f(\vec{x})$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$$

$$= \lim_{r\to 0} \ln\left(\frac{3r^2(\cos^2\theta + \sin^2\theta) - r^4\cos^2\theta\sin^2\theta}{r^2(\cos^2\theta+\sin^2\theta)}\right)$$

$$= \lim_{r\to 0} \ln(3 - r^2\cos^2\theta\sin^2\theta) = \ln 3$$

$\therefore f(0,0) := \ln 3$

Partial Derivatives (§ 14.3)

$f: \mathbb{R}^n \to \mathbb{R}$,

$$\frac{\partial f}{\partial x_i}(\vec{x}) := \lim_{h \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(\vec{x})}{h}$$

Notation:

- $f_x = \frac{\partial f}{\partial x} = \partial_x f$
- $f_y = \frac{\partial f}{\partial y} = \partial_y f$
- $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

  $= \partial_y(\partial_x f) = \partial_{yx} f$
- $f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

  $= \partial_x(\partial_y f) = \partial_{xy} f$
- $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

  $= \partial_x(\partial_x f) = \partial_{xx} f = \partial_x^2 f$

Similar for higher order / dimension.

Q 22). Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ of:

$$f(x,y) = \sum_{n=0}^{\infty} (xy)^n \quad , \quad \text{on } |xy| < 1$$

then
$$\sum_{n=0}^{\infty} (xy)^n = \frac{1}{1-xy}$$

$$\frac{\partial f}{\partial x} = y(1-xy)^{-2}$$

$$\frac{\partial f}{\partial y} = x(1-xy)^{-2}$$

Q 52). Show $w_{xy} = w_{yx}$

where $w = e^x + x\ln y + y\ln x$

$$w_x = e^x + \ln y + \frac{y}{x}$$
$$w_{xy} = \frac{1}{y} + \frac{1}{x}$$

$$w_y = \frac{x}{y} + \ln x$$
$$w_{yx} = \frac{1}{y} + \frac{1}{x}$$

Q60) Use definition to compute:

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)}, \quad \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$$

where $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h^3)}{h^3}$$

$$= 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h^4)}{h^3}$$

$$= \lim_{h\to 0} \frac{\sin(h^4)}{h^4} \cdot h$$

$$= 0.$$